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On the relation between the wave aberration function and the phase transfer function for an incoherent imaging system with circular pupil

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Abstract. The consideration is carried out in its general formulation: the wave aberration function is represented in terms of classical aberrations (the Zernike polynomials), the phase transfer function (argument of the complex optical transfer function) is defined by a chain of transformations originating from the generalized pupil function. Quasi-analytical quadrature formulas are derived that link the optical transfer function and the phase transfer function with the aberration terms. It is shown that the phase transfer function contains information on the odd-order aberrations, which can be retrieved from coefficients to the Taylor expansion of the derived quadrature relation.

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Rapid Note

The phase distortion of an initially plane transverse reference wave by an optical system can be quantitatively represented by the *wave aberration function* $\Psi(x,y)$ describing the phase distribution at the exit pupil plane XY. In the presence of wave aberrations the *generalized pupil function* $P(x,y)$ is characterized by the complex value

$$P(x,y) = A(x,y)\,\mathrm{e}^{\mathrm{i}\Psi(x,y)}, \tag{1}$$

where $\mathrm{i} = \sqrt{-1}$ and the *pupil function* $A(x,y)$ is unity inside and zero outside the pupil aperture. In the image plane $X_{\mathrm{im}}Y_{\mathrm{im}}$ the distorted wavefront produces the *coherent impulse response* [1]

$$H_{\mathrm{c}}(x_{\mathrm{im}},y_{\mathrm{im}}) = \frac{C}{\lambda z}\iint_{-\infty}^{+\infty} P(x,y)\,\mathrm{e}^{-\mathrm{i}\frac{2\pi}{\lambda z}(x_{\mathrm{im}}x+y_{\mathrm{im}}y)}\mathrm{d}x\mathrm{d}y, \tag{2}$$

where λ is the operating wavelength of a (monochromatic) image detector, z is the distance between the exit pupil and the image plane, and C is a constant amplitude. In this paper we discuss an imaging system that uses incoherent illumination. Such systems are characterized by the *incoherent impulse response*

$$H(x_{\mathrm{im}},y_{\mathrm{im}}) = H_{\mathrm{c}}(x_{\mathrm{im}},y_{\mathrm{im}})\,H_{\mathrm{c}}^{*}(x_{\mathrm{im}},y_{\mathrm{im}}). \tag{3}$$

The *optical transfer function* (OTF) is defined in the space of the spatial frequencies u and v along x and y axes as a Fourier image of the incoherent impulse response

$$H_{\mathrm{f}}(u,v) = \frac{\iint_{-\infty}^{+\infty} H(x_{\mathrm{im}},y_{\mathrm{im}})\,\mathrm{e}^{-\mathrm{i}2\pi(x_{\mathrm{im}}u+y_{\mathrm{im}}v)}\mathrm{d}x_{\mathrm{im}}\mathrm{d}y_{\mathrm{im}}}{\iint_{-\infty}^{+\infty} H(x_{\mathrm{im}},y_{\mathrm{im}})\,\mathrm{d}x_{\mathrm{im}}\mathrm{d}y_{\mathrm{im}}}. \tag{4}$$

This function can be represented in the modulus-argument notation as follows

$$H_{\mathrm{f}}(u,v) = B(u,v)\,\mathrm{e}^{\mathrm{i}\Phi(u,v)}, \quad B(u,v) = |H_{\mathrm{f}}(u,v)|,$$
$$\Phi(u,v) = \arg(H(u,v)), \tag{5}$$

where the real functions $B(u,v)$ and $\Phi(u,v)$ are referred to as the *modulation transfer function* and the *phase transfer function* respectively.

The problem of partial restoration of the wave aberration function from some measured characteristics of the OTF is one of the central issues of the Fourier and adaptive optics. Today's researches involve computer-aided techniques [2–4] and investigation of specific cases (for example, graded-index optical systems [5]) accompanied by the introduction of new conceptions, such as the pseudo modulation transfer function [6], and novel mathematical methods, *e.g.*, wavelet and time-frequency analysis [7]. In the present work we will focus on the poorly examined phase transfer function, rather than on the OTF and the modulation transfer function. Apart from its own scientific interest, this research is stimulated by our recent investigations [8] that indicate the possibility to retrieve,

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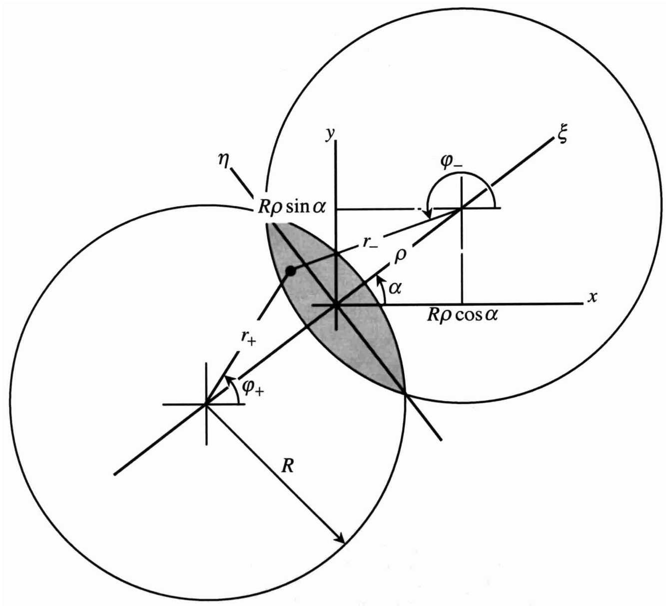

Fig. 1. Calculation of the optical transfer function: coordinate systems, integration domain, and parameters.

in some particular cases, the phase transfer function entirely *via* the statistical analysis of the spatial spectra of a sufficiently large set of images, without any additional equipment and measurements.

We will define what information about the wave aberrations is still retained in the phase transfer function after the transformation chain (1–5) starting from the explicit relation between $H_\mathrm{f}(u,v)$ and $P(x,y)$, equivalent to that discussed in Goodman's book [1],

$$H_\mathrm{f}(u,v) = \frac{1}{S_\mathrm{p}} \iint_{-\infty}^{+\infty} P\left(x - \frac{1}{2}\lambda z u,\ y - \frac{1}{2}\lambda z v\right) \times P^*\left(x + \frac{1}{2}\lambda z u,\ y + \frac{1}{2}\lambda z v\right) \mathrm{d}x\mathrm{d}y \quad (6)$$

that can be obtained by direct substitution of equations (3) and (2) into definition (4). Here S_p is the area of the exit pupil.

From here on we will discuss a circular pupil of radius R, for which

$$A(x,y) = h\left(1 - \frac{\sqrt{x^2+y^2}}{R}\right), \quad h(\varsigma) = \begin{cases} 0 & \varsigma < 0 \\ 1 & \varsigma \geq 0 \end{cases},$$

and actual domain of integration is the intersection of two circles shown in Figure 1. We first express the OTF *via* the frequency-domain dimensionless polar coordinates ρ, α

$$u = \frac{2R}{\lambda z}\rho\cos\alpha, \qquad v = \frac{2R}{\lambda z}\rho\sin\alpha \quad (7)$$

and, then, pass to the dimensionless Cartesian coordinates

$$\xi = \frac{x}{R}\cos\alpha + \frac{y}{R}\sin\alpha, \quad \eta = -\frac{x}{R}\sin\alpha + \frac{y}{R}\cos\alpha, \quad (8)$$

which coincide with the axes of symmetry of the integration domain. Using this symmetry, one can represent $H_\mathrm{f}(u,v)$ *via* its polar-coordinate counterpart $H_\mathrm{fp}(\rho,\alpha)$ as follows

$$\begin{aligned} H_\mathrm{f}(u,v) &= H_\mathrm{fp}\left(\rho = \frac{\lambda z}{2R}\sqrt{u^2+v^2}, \alpha = \arg(u+\mathrm{i}v)\right) \\ &= \frac{1}{\pi}\int_0^{\sqrt{1-\rho^2}} \int_{-\left(\sqrt{1-\eta^2}-\rho\right)}^{\sqrt{1-\eta^2}-\rho} \left(\mathrm{e}^{\mathrm{i}[\Psi_\alpha(\xi,-\rho,\eta)-\Psi_\alpha(\xi,\rho,\eta)]}\right. \\ &\quad \left. + \mathrm{e}^{-\mathrm{i}[\Psi_\alpha(-\xi,\rho,-\eta)-\Psi_\alpha(-\xi,-\rho,-\eta)]}\right) \mathrm{d}\xi\mathrm{d}\eta, \end{aligned} \quad (9)$$

where

$$\begin{aligned} \Psi_\alpha(\xi,\pm\rho,\eta) = \Psi(x &= [(\xi\pm\rho)\cos\alpha - \eta\sin\alpha]\,R, \\ y &= [(\xi\pm\rho)\sin\alpha + \eta\cos\alpha]\,R) \end{aligned} \quad (10)$$

and $0 \leq \rho \leq 1$, as for the area $\rho > 1$ one has $H_\mathrm{fp} \equiv 0$.

Let us follow the traditional way of analysis of the wave distortion by its decomposition into a set of classic aberration terms — the Zernike polynomials $Z_n^m(r,\varphi)$ [9]

$$\Psi(x,y) = \Psi_\mathrm{p}(r,\varphi) = \sum_{n,m} \Psi_{n,m} Z_n^m(r,\varphi), \quad (11)$$

where $\Psi_\mathrm{p}(r,\varphi)$ is the representation of $\Psi(x,y)$ in the polar coordinates $r = \sqrt{x^2+y^2}/R$, $\varphi = \arg(x+\mathrm{i}y)$, $n \leq 0$, $-n \leq m \leq n$, $n-|m|$ is even, and

$$Z_n^m(r,\varphi) = \begin{cases} R_n^m(r)\cos(m\varphi) & m \geq 0 \\ R_n^{|m|}(r)\sin(m\varphi) & m < 0 \end{cases},$$

$$R_n^m(r) = \sum_{k=0}^{(n-m)/2} \frac{(-1)^k(n-k)!}{k!\left(\frac{n+m}{2}-k\right)!\left(\frac{n-m}{2}-k\right)!} r^{n-2k}. \quad (12)$$

This allows us to rewrite expansion (11) explicitly as a sum of even and odd aberrations $\Psi_\mathrm{p}(r,\varphi) = V_\mathrm{e}(r,\varphi) + V_\mathrm{o}(r,\varphi)$, where

$$\begin{aligned} V_\mathrm{e}(r,\varphi) &= \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \Psi_{2n,2m} Z_{2n}^{2m}(r,\varphi), \\ V_\mathrm{o}(r,\varphi) &= \sum_{n=0}^{\infty}\sum_{m=-n-1}^{n} \Psi_{2n+1,2m+1} Z_{2n+1}^{2m+1}(r,\varphi). \end{aligned} \quad (13)$$

A principal point of our consideration is that, apart from the explicit formulas for the Zernike polynomials, the polar-coordinate representation r, φ gives us one more serious advantage, namely, the phase components in the integrand of the OTF representation (9) are expressed in a shorter two-argument form

$$\begin{aligned} &\Psi_\alpha(\xi,-\rho,\eta) = \Psi_\mathrm{p}(r_-,\varphi_-), \quad \Psi_\alpha(\xi,\rho,\eta) = \Psi_\mathrm{p}(r_+,\varphi_+), \\ &\Psi_\alpha(-\xi,\rho,-\eta) = \Psi_\mathrm{p}(r_-,\varphi_- + \pi), \\ &\Psi_\alpha(-\xi,-\rho,-\eta) = \Psi_\mathrm{p}(r_+,\varphi_+ + \pi), \end{aligned}$$

where $r_\pm = \sqrt{(\xi \pm \rho)^2 + \eta^2}$, $\varphi_\pm = \arg(\xi \pm \rho + \mathrm{i}\eta) + \alpha$. Noticing that $V_\mathrm{e}(r_\pm, \varphi_\pm + \pi) = V_\mathrm{e}(r_\pm, \varphi_\pm)$ and $V_\mathrm{o}(r_\pm, \varphi_\pm + \pi) = -V_\mathrm{o}(r_\pm, \varphi_\pm)$, we get polar-coordinate representations of the OTF and the phase transfer function *via* the classic aberration terms, in which the integrand has some similarity with signals of the lateral sharing interferometry [10]

$$H_\mathrm{fp}(\rho,\alpha) = \frac{2}{\pi}\int_0^{\sqrt{1-\rho^2}} \int_{-\left(\sqrt{1-\eta^2}-\rho\right)}^{\sqrt{1-\eta^2}-\rho} \cos\left(V_\mathrm{e}(r_+,\varphi_+) - V_\mathrm{e}(r_-,\varphi_-)\right) \mathrm{e}^{-\mathrm{i}[V_\mathrm{o}(r_+,\varphi_+)-V_\mathrm{o}(r_-,\varphi_-)]}\mathrm{d}\xi\mathrm{d}\eta, \quad (14)$$

$$\begin{aligned}\Phi(u,v) &= \Phi_\mathrm{p}\left(\rho = \frac{\lambda z}{2R}\sqrt{u^2+v^2}, \alpha = \arg(u+\mathrm{i}v)\right)\\ &= \arg\left(\int_0^{\sqrt{1-\rho^2}} \int_{-\left(\sqrt{1-\eta^2}-\rho\right)}^{\sqrt{1-\eta^2}-\rho} \cos\left(V_\mathrm{e}(r_+,\varphi_+) - V_\mathrm{e}(r_-,\varphi_-)\right) \mathrm{e}^{-\mathrm{i}[V_\mathrm{o}(r_+,\varphi_+)-V_\mathrm{o}(r_-,\varphi_-)]}\mathrm{d}\xi\mathrm{d}\eta\right). \quad (15)\end{aligned}$$

For further analysis let us turn to representation of Φ_p through the Taylor series in the vicinity of $\rho = 1$. Introducing the expansion parameter $\varepsilon = \sqrt{1-\rho^2}$ and changing the variables $\eta \to \varepsilon\mu$, $\xi \to \varepsilon^2\gamma$, we get

$$\Phi_\mathrm{p}(\rho,\alpha) = \arg\left(\int_0^1 \int_{-\gamma_0}^{\gamma_0} \cos\left(V_\mathrm{e}(r_+,\varphi_+) - V_\mathrm{e}(r_-,\varphi_-)\right) \times \exp\left(-\mathrm{i}\left[V_\mathrm{o}(r_+,\varphi_+) - V_\mathrm{o}(r_-,\varphi_-)\right]\right)\mathrm{d}\gamma\mathrm{d}\mu\right), \quad (16)$$

where

$$\gamma_0 = \frac{\sqrt{1-(\varepsilon\mu)^2} - \sqrt{1-\varepsilon^2}}{\varepsilon^2},$$

$$r_\pm = \sqrt{\left(\varepsilon^2\gamma \pm \sqrt{1-\varepsilon^2}\right)^2 + \varepsilon^2\mu^2},$$

$$\varphi_\pm = \arg\left(\varepsilon^2\gamma \pm \sqrt{1-\varepsilon^2} + \mathrm{i}\varepsilon\mu\right) + \alpha.$$

In the domain $1/2 \le \rho \le 1$, which corresponds to $0 \le \varepsilon \le \sqrt{3}/2$, parameters $\varphi_\pm$ can be represented in the form

$$\varphi_+ = \arctan\left(\frac{\varepsilon\mu}{\sqrt{1-\varepsilon^2}+\varepsilon^2\gamma}\right) + \alpha,$$

$$\varphi_- = \pi - \arctan\left(\frac{\varepsilon\mu}{\sqrt{1-\varepsilon^2}-\varepsilon^2\gamma}\right) + \alpha, \quad (17)$$

and application of the expansion procedure with respect to ε yields

$$\begin{aligned}\Phi_\mathrm{p}(\rho,\alpha) &= -2V_\mathrm{o} + \frac{1}{5}\left(4\frac{\partial V_\mathrm{o}}{\partial r} - \frac{\partial^2 V_\mathrm{o}}{\partial \varphi^2}\right)(1-\rho^2)\\ &+ \frac{1}{700}\left(4\left[16\left(\frac{\partial V_\mathrm{e}}{\partial\varphi}\right)^2 - 27\right]\frac{\partial^2 V_\mathrm{o}}{\partial\varphi^2} - 5\frac{\partial^4 V_\mathrm{o}}{\partial\varphi^4}\right.\\ &\left.+8\left\{\left[14 + 8\left(\frac{\partial V_\mathrm{e}}{\partial\varphi}\right)^2\right]\frac{\partial V_\mathrm{o}}{\partial r} + 5\frac{\partial^3 V_\mathrm{o}}{\partial r\partial\varphi^2} - 20\frac{\partial^2 V_\mathrm{o}}{\partial r^2}\right\}\right)\\ &\times\left(1-\rho^2\right)^2 + O\left(\left(1-\rho^2\right)^3\right). \quad (18)\end{aligned}$$

Here the right-hand value should be taken for $r = 1$ and $\varphi = \alpha$.

General quadrature formula (15) and expansion (18) do not provide an explicit relation between the phase transfer function and the wave aberration function. However, the results obtained enable us to make the following remarks and conclusions.

(i) Although the phase transfer function contains information about both even and odd aberration terms, data on the even component of the wave aberration function can be hardly obtained. In the absence of the odd-order aberrations ($V_\mathrm{o} \equiv 0$) this is easily seen as equation (14) yields purely real value of the OTF (the same can be demonstrated with initial expression (6) provided that the exit pupil possesses enough symmetry). A less trivial result is that even if the odd-order aberrations are significant, information about the even-order ones is deeply hidden in modulation factors of higher-order terms of the expansion (18) and, in view of remark (iii) below, is practically unavailable.

(ii) If the phase transfer function can be retrieved *via* the image analysis, one gets an opportunity of at least some odd-order aberration assessment for granted, which is of primary importance for imaging systems operating in autonomous conditions. Estimation of aberrations due to the inhomogeneous thermal load of sun radiation in the space-borne systems of earth observation [8] is an example.

(iii) The efficiency of methods based on the above quasi-analytical expansion is seriously limited by the noise factors. Their influence increases with the term order as successively higher derivatives are involved in more and more complicated algebraic structures. Terms of the order $\left(1-\rho^2\right)^3$ and higher are of doubtful practical significance; the three-term expansion (18) corresponds to limiting cases of excellent image detection, wherein one deals with cumulative data and stable, long-term aberrations.

(iv) Concrete methods of inversion of the obtained relationship and analysis of their efficiency require consideration of a particular imaging train, detecting system, data-processing scheme (with or without cumulative data), range of possible signals, statistical properties of the noise, etc. This consideration goes outside the scope of this *Rapid Note* and will be published elsewhere. In most cases the phase-restoration technique in question is limited by noise to a few first aberrations, such as tilt, lateral coma, and three-leaf clover (but it is these aberrations that are of primary importance in many problems of adaptive optics).

(v) Even remaining within the framework of the general consideration given above, we can suggest the algorithm of quasi-analytical restoration to the odd-order aberrations, in which the application of an interpolation procedure, which provides us the Taylor expansion coefficients with "differential" methods and thus inevitably leads to the noise amplification, is compensated by integration (averaging) with respect to the polar angle at the final stage of separation of the aberration terms. For the sake of brevity we will constrain the restoration accuracy to the aberrations listed in remark (iv), which is equivalent to representation of the wave aberration function *via* the first ten Zernike polynomials

$$\begin{aligned} V_{\mathrm{e}}(r,\varphi) &= \Psi_{0,0} + \Psi_{2,-2} r^2 \sin 2\varphi + \Psi_{2,0}\left(2r^2-1\right) \\ &\quad + \Psi_{2,2} r^2 \cos 2\varphi, \\ V_{\mathrm{o}}(r,\varphi) &= \Psi_{1,-1} r \sin\varphi + \Psi_{1,1} r\cos\varphi + \Psi_{3,-3} r^3 \sin 3\varphi \\ &\quad + \Psi_{3,-1}\left(3r^3-2r\right)\sin\varphi + \Psi_{3,1}\left(3r^3-2r\right)\cos\varphi \\ &\quad + \Psi_{3,3} r^3 \cos 3\varphi. \end{aligned}$$

Then restoration of the odd-order aberration coefficients $\Psi_{1,\pm1}$, $\Psi_{3,\pm1}$, and $\Psi_{3,\pm3}$ from the set of given values of $\Phi(u,v)$ is achievable within the following three steps.

(1) Polar-coordinate representation. The dimensionless polar coordinate system $\varepsilon \in [0,1]$, $\alpha \in [0,\pi]$ is introduced,

$$\begin{aligned} \varepsilon &= \sqrt{1-\rho^2} = \sqrt{1-\left(\frac{\lambda z}{2R}\right)(u^2+v^2)}, \\ \alpha &= \arg(u + \mathrm{i}v), \end{aligned}$$

and the set of measured values of the phase transfer function is represented in this coordinate system

$$\begin{aligned} \Phi_\varepsilon(\varepsilon,\alpha) &= \Phi_{\mathrm{p}}\left(\rho=\sqrt{1-\varepsilon^2},\alpha\right) \\ &= \Phi\left(u=\frac{2R}{\lambda z}\sqrt{1-\varepsilon^2}\cos\alpha,\ v=\frac{2R}{\lambda z}\sqrt{1-\varepsilon^2}\sin\alpha\right). \end{aligned} \tag{19}$$

(2) Taylor expansion. Using some interpolation procedures (see, for example, [12]), the obtained set is converted into the Taylor series with respect to ε^2

$$\Phi_\varepsilon(\varepsilon,\alpha) = \Phi_0(\alpha) + \Phi_1(\alpha)\varepsilon^2 + \Phi_2(\alpha)\varepsilon^4 + \ldots \tag{20}$$

that gives us functions Φ_0 and Φ_1 of only one argument. Comparing representations (18) and (20) we get

$$\begin{aligned} \Phi_0(\alpha) &= -2V_0(1,\alpha), \\ \Phi_1(\alpha) &= \frac{1}{5}\left(4\frac{\partial V_{\mathrm{o}}}{\partial r}(1,\alpha) - \frac{\partial^2 V_{\mathrm{o}}}{\partial\varphi^2}(1,\alpha)\right). \end{aligned} \tag{21}$$

(3) Separation of the aberration terms. Applying the integral operators $\frac{1}{\pi}\int_0^{2\pi}(*)\begin{pmatrix}\sin 3\alpha\\ \cos 3\alpha\end{pmatrix}\mathrm{d}\alpha$ to $\Phi_0(\alpha)$ and $\frac{1}{\pi}\int_0^{2\pi}(*)\begin{pmatrix}\sin \alpha\\ \cos \alpha\end{pmatrix}\mathrm{d}\alpha$ to both $\Phi_0(\alpha)$ and $\Phi_1(\alpha)$ we get

$$\begin{aligned} &\Phi_{0,\pm3} = -2\Psi_{3,\pm3}, \quad \Phi_{0,\pm1} = -2\left(\Psi_{1,\pm1}+\Psi_{3,\pm1}\right), \\ &\Phi_{1,\pm1} = \frac{1}{5}\left(5\Psi_{1,\pm1} + 29\Psi_{3,\pm1}\right), \end{aligned} \tag{22}$$

where

$$\begin{aligned} \Phi_{0,\pm3} &= \frac{1}{\pi}\int_0^{2\pi}\Phi_0(\alpha)\begin{pmatrix}\cos 3\alpha\\ \sin 3\alpha\end{pmatrix}\mathrm{d}\alpha, \\ \Phi_{k,\pm1} &= \frac{1}{\pi}\int_0^{2\pi}\Phi_k(\alpha)\begin{pmatrix}\cos \alpha\\ \sin \alpha\end{pmatrix}\mathrm{d}\alpha, \quad k=0,1. \end{aligned} \tag{23}$$

Solving the system of linear equations (22) yields a representation of the desired odd-order aberration coefficients *via* the data retrieved from the phase transfer function

$$\begin{aligned} \Psi_{1,\pm1} &= -\frac{1}{48}\left(29\Phi_{0,\pm1} + 10\Phi_{1,\pm1}\right), \\ \Psi_{3,\pm1} &= -\frac{5}{48}\left(\Phi_{0,\pm1} + 2\Phi_{1,\pm1}\right), \quad \Psi_{3,\pm3} = -\frac{1}{2}\Phi_{0,\pm3}. \end{aligned}$$

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